UDC 531.36

ON INSTABILITY OF THE POSITION OF EQUILIBRIUM OF A HAMILTONIAN SYSTEM

PMM Vol.42, № 3, 1978, pp.557-562

M.A. BALITINOV
(Makhachkala)

(Received December 21, 1976)

Consider the following system of Hamiltonian equations:

$$q_j^{\cdot} = \frac{\partial H}{\partial p_j}, \quad p_j^{\cdot} = -\frac{\partial H}{\partial q_j}, \quad j = 1, 2, \ldots, n \tag{1}$$

$$H(\mathbf{q}, \mathbf{p}) = T + V = \frac{1}{2} \sum_{i, j=1}^{n} b_{ij}(\mathbf{q}) p_i p_j + V(\mathbf{q})$$

We assume that T is a positive definite quadratic form near the point $\mathbf{q} = 0$ and the origin is an isolated state of equilibrium of the system (1).

Below we obtain the sufficient conditions of instability of the position of equilibrium of the system (1), which generalizes certain results already obtained (see [1, 4] and others). In what follows, we shall assume that

$$\sum_{i, j=1}^{n} b_{ij}(0) p_i p_j = \sum_{j=1}^{n} p_j^2$$

since otherwise we can attain it by means of the linear canonical transformation $\mathbf{q} = D'\mathbf{x}$, $\mathbf{y} = D\mathbf{p}$, where $B = \{b_{ij}(0)\} = DD'$ with the generating function $W(\mathbf{x}, \mathbf{p}) = x'D\mathbf{p}$.

Theorem 1. Let the following conditions hold:

1°. The potential energy can be written in the form of a sum $V(\mathbf{q}) = V_\mu(\mathbf{q}) + \Phi(\mathbf{q})$, where $V_\mu(\mathbf{q})$ is a homogeneous function of dimension $\mu \geqslant 2$.

2°. $V_\mu(\mathbf{q}) \in c^{(2)}$, $\Phi(\mathbf{q}) \in c^{(2)}$, $b_{ij}(\mathbf{q}) \in c^{(1)}$, $\min_{|\mathbf{q}|=1} V_\mu(q) = V_\mu(c) = -\lambda < 0$.

3°. For sufficiently small τ and $\mathbf{z}$ (δ is an arbitrarily small quantity) $|F_i(\tau, 0)| \leqslant N|\tau|$,

$$|F_i(\tau, \mathbf{z}_{**}) - F_i(\tau, \mathbf{z}_*)| \leqslant \delta |\mathbf{z}_{**} - \mathbf{z}_*|,$$

where

$$F_i(\tau, \mathbf{z}) = \tau^{1-\mu} \left. \frac{\partial \Phi}{\partial q_i} \right|_{qj = \tau(c_j + z_{j-1})}$$

Then the zero solution of the system (1) is unstable.

Proof. It is sufficient to show that the system (1) has a solution $\mathbf{p}(t), \mathbf{q}(t)$ with the property that $\mathbf{p}(t) \to 0$ and $\mathbf{q}(t) \to 0$ as $t \to -\infty$ or $t \to +\infty$ (obviously, in addition to the solution $\mathbf{p}(t), \mathbf{q}(t)$ the system (1) also has a solution $-\mathbf{p}(-t), \mathbf{q}(-t)$).

Let us perform the change of variables

$$q_1 = c_1\tau, \quad q_i = \tau(c_i + z_{i-1}), \quad i = 2, 3, \ldots, n$$
$$p_i = (2\lambda)^{1/2} \tau^{1/2\mu}(c_i + z_{n-1+i}), \quad i = 1, 2, \ldots, n \tag{2}$$

We assume that $c_1 \neq 0$, since otherwise q_i could be renumbered.
From (2) we obtain

$$\tau \frac{dz_{i-1}}{d\tau} = \frac{dq_i}{d\tau} - c_i - z_{i-1} = c_1 \left(\frac{dq_1}{dt}\right)^{-1} \frac{dq_i}{dt} - c_i - z_{i-1} =$$

$$c_1 \left[\sum_{j=1}^{n} b_{1j}(\mathbf{q}) p_j\right]^{-1} \sum_{j=1}^{n} b_{ij}(\mathbf{q}) p_j - c_i - z_{i-1} = - z_{i-1} - c_1^{-1} c_i z_n +$$

$$z_{n-1+i} + \ldots, \quad i = 2, 3, \ldots, n$$

$$\tau \frac{dz_{n-1+i}}{d\tau} = (2\lambda)^{-1/2} \tau^{1-1/2\mu} \frac{dp_i}{d\tau} - \frac{1}{2} \mu (c_i + z_{n-1+i}) =$$

$$\frac{1}{2} c_1 (2\lambda)^{-1/2} \tau^{1-1/2\mu} \left[\sum_{j=1}^{n} b_{1j}(\mathbf{q}) p_j\right]^{-1} \left[\sum_{k, j=1}^{n} \frac{\partial b_{kj}}{\partial q_i} p_k p_j + \frac{\partial V}{\partial q_i}\right] -$$

$$\frac{1}{2} \mu (c_i + z_{n-1+i}) = \sum_{j=1}^{2n-1} d_{ij} z_j + \ldots$$

where the repeated dots denote either the terms containing τ, or those of order higher than the first in z_j. In deriving the above expressions, we made use of the equations

$$\sum_{j=1}^{n} b_{1j}(\mathbf{q}) p_j = p_1 + \ldots = (2\lambda)^{1/2} \tau^{1/2\mu} (c_1 + z_n) + \ldots$$

$$\left.\frac{\partial V_\mu}{\partial q_j}\right|_{q_k = c_k} = - \lambda \mu c_j, \quad \left.\frac{\partial (V_\mu + \Phi)}{\partial q_j}\right|_{q_k = \tau(c_k + z_{k-1})} =$$

$$\tau^{\mu-1} \left(\left.\frac{\partial V_\mu}{\partial q_j}\right|_{q_k = c_k} + \sum_{i=1}^{n} \left.\frac{\partial^2 V_\mu}{\partial q_j \partial q_i} z_i \right|_{q_k = c_k} + \ldots \right) + \left.\frac{\partial \Phi}{\partial q_j}\right|_{q_k = \tau(c_k + z_{k-1})}$$

which follow from the conditions of Theorem 1. As the result we have

$$\tau \frac{d\mathbf{z}}{d\tau} = A\mathbf{z} + f(\tau, \mathbf{z}) \tag{3}$$

Here A is a constant $(2n-1)$ $(2n-1)$ matrix and the vector function $f(\tau, \mathbf{z})$ satisfies, in sufficiently small neighborhood of the coordinate origin, the following conditions:

$$|f(\tau, 0)| \leqslant N_* |\tau|, \quad |f(\tau, \mathbf{z}_{**}) - f(\tau, \mathbf{z}_*)| \leqslant \varepsilon |\mathbf{z}_{**} - \mathbf{z}_*|, \quad N_* = \text{const}$$

(ε is an arbitrarily small quantity).

We know that when the above conditions hold, the system (3) has at least one trajectory emerging from the coordinate origin. Let $z_j = \varphi_j(\tau)$ $(j = 1, 2, \ldots, 2n-1)$ be a solution of the system (3) and let $\varphi_j(\tau) \to 0$ as $\tau \to +0$. Then

$$q_1 = c_1 \tau, \quad q_i = \tau (c_i + \varphi_{i-1}(\tau)), \quad i = 2, 3, \ldots, n$$

$$p_i = (2\lambda)^{1/2} \tau^{1/2\mu} (c_i + \varphi_{n-1+i}(\tau)), \quad i = 1, 2, \ldots, n$$

is the phase trajectory of the system (1) adjacent to the position of equilibrium. It follows therefore that the system (1) has a phase trajectory along which the solutions attain the position of equilibrium as $t \to -\infty$. This in turn implies the instability of the position of equilibrium.

Let us give a geometrical representation of the conditions of Theorem 1. Consider the systems of differential equations with the corresponding Hamiltonians

$$H_1 = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + V_\mu(\mathbf{q}), \quad H_2 = \frac{1}{2}\sum_{i,j=1}^{n} b_{ij}(\mathbf{q})\, p_i p_j + V_\mu(\mathbf{q}) + \Phi(\mathbf{q})$$

The system corresponding to the Hamiltonian H_1 has the solution

$$q_j = c_j\tau, \quad p_j = c_j\tau, \; \tau = \begin{cases} \exp\left[(2\lambda)^{1/2} t\right], & \mu = 2 \\ \left[1 + \frac{2-\mu}{2}(2\lambda)^{1/2} t\right]^{2/(2-\mu)}, & \mu > 2 \end{cases}$$

where the vector $\mathbf{c}' = (c_1, c_2, \ldots, c_n)$ and the number λ are determined from the condition

$$\min_{|q|=1} V_\mu(\mathbf{q}) = V_\mu(\mathbf{c}) = -\lambda, \quad \lambda > 0$$

Consequently the relations $\partial V_\mu(c_1, c_2, \ldots, c_n) / \partial c_j + \lambda \mu c_j = 0$ hold.

The particular solution obtained shows that the position of equilibrium of this system is unstable. The conditions of the theorem ensure the proximity of the systems with the Hamiltonians H_1 and H_2 along the curve $q_j = c_j\tau$, $p_j = (2\lambda)^{1/2}\tau^{1/2}\mu c_j$ which represents the phase trajectory of the unperturbed system.

When the function $V_\mu(\mathbf{q})$ is nonnegative and $V(\mathbf{q})$ may assume negative values near the stationary point, then the theorem proved above can no longer be applied to the system (1). In this case the Chetaev theorem (see [2]) is found useful. However, Chetaev did not give any general examples of constructing the vector function $\mathbf{F} = (F_1, F_2, \ldots, F_n)$, which appears in the conditions of the theorem.

We shall now show a method of constructing the vector function in question by means of several examples.

Example 1. Consider a system with the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + V(\mathbf{q}) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \frac{1}{2}\sum_{j=2}^{n} c_j q_j^2 + V_3 + V_4 + \ldots$$

Let $c_i > 0$ $(i = 2, 3, \ldots, n)$ and the function $V(\mathbf{q})$ be of alternating sign in any neighborhood of the coordinate origin. We seek the vector function $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ as a solution of the equation

$$F_1 V_{q_1} + F_2 V_{q_2} + \ldots + F_n V_{q_n} = V \tag{4}$$

in the form

$$F_1 = F_1(q_1), \quad F_2 = F_2(q_1, q_2), \ldots, \quad F_n = F_n(q_1, q_2, \ldots, q_n)$$

Let $q_j = \varphi_j(q_1)$, $j = 2, \ldots, n$ be a solution of the system $V_{q_2} = 0$, $V_{q_3} = 0, \ldots,$ $V_{q_n} = 0$ (according to known theorems the system in question has a unique analytic solution). Considering the equation (4) along the curve $q_j = \varphi_j(q_1)$, we obtain

$$F_1(q_1) = V(V_{q_1})^{-1}\big|_{q_j=\varphi_j(q_1)} = f(q_1)\left[f'(q_1)\right]^{-1} = \sum_{j=1}^{\infty} \alpha_j q_1^j, \quad \alpha_1 = m^{-1}$$

$(f(q_1) \not\equiv 0$, since at a fixed value of q_1 the function $V(q_1, \ldots, q_n)$ assumes its minimum value at $q_j = \varphi_j(q_1)$ and this value is by definition negative). To find $F_2(q_1, q_2)$ we consider the equation (4) along the unique analytic solution $q_j = \psi_j(q_1, q_2)$ $(j = 3, 4, \ldots,)$ of the system $V_{q_3} = 0, \ldots, V_{q_n} = 0$. We obtain

$$F_2(q_1, q_2) = [(V - F_1 V_{q_1}) V_{q_2}^{-1}]\,|_{q_j=\psi_j}$$

Using the Weierstrass theorem [6] we can show that $F_2(q_1, q_2)$ is an analytic function, and we obviously have $F_2(q_1, q_2) = {}^1/_2 q_2 + \ldots$.

In this manner we determine, one after the other $F_j = {}^1/_2 q_j + \ldots$. In what follows, we shall find useful the following lemma.

Lemma. Let $V(x, y) = V_m + V_{m+1} + \ldots$, and let the following conditions hold:

1°. Equation $V_y = 0$ defines the functions $y = \theta_j(x) = \alpha_j x + \ldots$ They have, in general, complex coefficients and $\alpha_i \neq \alpha_j$ if $i \neq j$, $\alpha_j \neq 0$, $j = 1, 2, \ldots, m-1$.

2°. $V(x, \theta_j(x)) = \gamma_j^{(0)} x^m + \gamma_j^{(1)} x^{m+1} + \ldots$, $\gamma_j^{(0)} \neq 0$. In this case an analytic vector function $F = (F_1, F_2)$ exists which is a solution of the equation

$$F_1 V_x + F_2 V_y = (x^2 + y^2)^l\, V \tag{5}$$

$$l = \begin{cases} {}^1/_2 m - 2, & \text{for even } m \\ {}^1/_2(m+1) - 2, & \text{for odd } m \end{cases}$$

Proof. We seek a solution of (5) in the form

$$F_1(x, y) = \psi_0(x) + \psi_1(x)\, y + \ldots + \psi_{m-2}(x)\, y^{m-2}, \qquad F_2 = F_2(x, y)$$

Considering Eq. (5) for $y = \theta_j(x)$ $(j = 1, 2, \ldots, m-1)$, we obtain the following system for determining $\psi_j(x)$:

$$\begin{aligned} &\psi_0 + \theta_1\psi_1 + \ldots + \theta_1^{m-2}\psi_{m-2} = (x^2 + \theta_1^2)^l\, \varphi_1(x) \\ &\psi_0 + \theta_2\psi_1 + \ldots + \theta_2^{m-2}\psi_{m-2} = (x^2 + \theta_2^2)^l\, \varphi_2(x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\psi_0 + \theta_{m-1}\psi_1 + \ldots + \theta_{m-1}^{m-2}\psi_{m-2} = (x^2 + \theta_{m-1}^2)^l\, \varphi_{m-1}(x) \end{aligned} \tag{6}$$

$$\varphi_j(x) = [V(V_x)^{-1}]\,|_{y=\theta_j(x)} = V(x, \theta_j(x)) \left(\frac{d}{dx} V(x, \theta_j(x))\right)^{-1} = m^{-1}x + \ldots$$

The functions $\psi_j(x)$ are obtained from (6) uniquely, and it can be shown that $\psi_j(x)$ are real analytic functions. Further we have

$$F_2 = [(x^2 + y^2)^l\, V(x, y) - F_1 V_x]\, V_y^{-1}$$

By virtue of the Weierstrass theorem mentioned above, $F_2(x, y)$ is an analytic function. Having found the lower terms of the analytic functions F_1 and F_2, we can show that

$$F_1 = m^{-1}x\,(x^2 + y^2)^l + \ldots, \qquad F_2 = m^{-1}y\,(x^2 + y^2)^l + \ldots$$

Example 2. Consider the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + V(\mathbf{q}) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=3}^{n} c_j q_j^2 + V_N + V_{N+1} + \ldots$$

$$c_j > 0, \quad j = 3, 4, \ldots, n, \quad N > 2$$

Let $q_j = \psi_j(q_1, q_2)$ $(j = 3, \ldots, n)$ be a solution of the system of equations $V_{q_3} = 0$, $V_{q_4} = 0, \ldots, V_{q_n} = 0$. We assume that the function

$$V(q_1, q_2, \psi_3, \ldots, \psi_n) = V^\circ(q_1, q_2) = V_m^\circ + V_{m+1}^\circ + \ldots$$

satisfies the conditions of the lemma. We seek the vector function $\mathbf{F} = (F_1, F_2, \ldots, F_n)$ as a solution of the equation

$$F_1 V_{q_1} + F_2 V_{q_2} + \ldots + F_n V_{q_n} = (q_1^2 + q_2^2)^l V \qquad (7)$$

in the form

$$F_1 = F_1(q_1, q_2), \quad F_2 = F_2(q_1, q_2), \quad F_3 = F_3(q_1, q_2, q_3), \ldots,$$
$$F_n = F_n(q_1, q_2, \ldots, q_n)$$

Considering now Eq. (7) on the manifold defined by the equations $q_j = \psi_j(q_1, q_2)$ $(j = 3, \ldots, n)$, we obtain (l is determined in the same manner as in the lemma)

$$F_1 V_{q_1}^\circ + F_2 V_{q_2}^\circ = (q_1^2 + q_2^2)^l V^\circ \qquad (8)$$

Equation (8) yields F_1 and F_2. Further, if $F_1, F_2, \ldots, F_k$ have been found, then F_{k+1} can be determined by considering Eq. (7) on the manifold defined by the equations $V_{q_{k+2}} = 0, \ldots, V_{q_n} = 0$. Clearly, F_j are analytic functions and their lower order terms have the form

$$F_1 = m^{-1} q_1 (q_1^2 + q_2^2)^l + \ldots, \quad F_2 = m^{-1} q_2 (q_1^2 + q_2^2)^l + \ldots$$

$$F_3 = {}^1/_2 q_3 (q_1^2 + q_2^2)^l + \ldots, \quad F_n = {}^1/_2 q_n (q_1^2 + q_2^2)^l + \ldots$$

The quadratic form

$$\sum_{i, j=1}^{n} g_{ij} p_i p_j, \quad g_{ij} = \frac{1}{2}\left(\frac{\partial F_i}{\partial q_j} + \frac{\partial F_j}{\partial q_i}\right)$$

is positive definite on the set

$$Q = Q_1 \cap Q_2; \quad Q_1 = \{V(\mathbf{q}) < 0\}, \quad Q_2 = \left\{\sum_{j=1}^{n} q_j^2 < h\right\}$$

(h is a sufficiently small number).

Indeed, we have $q_3^2 + q_4^2 + \ldots + q_n^2 < \delta (q_1^2 + q_2^2)$, on the set Q, where $\delta(h)$ is an arbitrarily small quantity when h is small. Writing the corresponding matrix in the form $G = G_0 + G_1 + \cdots$ where G_0 is a matrix composed of the lower order terms, we obtain

$$G_0 = r^{l-1} G_* = r^{l-1} \times$$

$$\times \left\| \begin{array}{cccccc} m^{-1}[(2l+1) q_1^2 + q_2^2] & 2lm^{-1} q_1 q_2 & {}^1/_2 l q_1 q_3 & \ldots & {}^1/_2 l q_1 q_{n-1} & {}^1/_2 l q_1 q_n \\ 2lm^{-1} q_2 q_1 & m^{-1}[q_1^2 + (2l+1) q_2^2] & {}^1/_2 l q_2 q_3 & \ldots & {}^1/_2 l q_2 q_{n-1} & {}^1/_2 l q_2 q_n \\ {}^1/_2 l q_3 q_1 & {}^1/_2 l q_3 q_2 & r & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ {}^1/_2 l q_{n-1} q_1 & {}^1/_2 l q_{n-1} q_2 & 0 & \ldots & r & 0 \\ {}^1/_2 l q_n q_1 & {}^1/_2 l q_n q_2 & 0 & \ldots & 0 & r \end{array} \right\|$$

$$(r = q_1^2 + q_2^2)$$

The Silvester determinants which begin in the lower right-hand corner of the matrix G_*, have the form

$$\Delta_1 = r, \ \Delta_2 = r^2, \ldots, \ \Delta_{n-2} = r^{n-2}$$

$$\Delta_{n-1} = r^{n-2} \left\{ m^{-1} [q_1^2 + (2l+1) q_2^2] - \frac{l^2 q_2^2}{4r} \rho \right\}$$

$$\Delta_n = \begin{vmatrix} m^{-1} [(2l+1) q_1^2 + q_2^2] - \frac{l^2 q_1^2}{4r} \rho & 2lm^{-1} q_1 q_2 - \frac{l^2 q_1 q_2}{4r} \rho \\ 2lm^{-1} q_1 q_2 - \frac{l^2 q_1 q_2}{4r} \rho & m^{-1} [q_1^2 + (2l+1) q_2^2] - \frac{l^2 q_2^2}{4r} \rho \end{vmatrix}$$

$(\rho = q_3^2 + q_4^2 + \ldots + q_n^2)$

In computing Δ_{n-1} and Δ_n, we have used the following obvious assumption : if Δ, c, b and k are the $m \times m$, $m \times n$, $n \times m$, $n \times n$ matrices respectively, and $|k| \neq 0$, then

$$\begin{vmatrix} \Delta & c \\ b & k \end{vmatrix} = |k| \, |\Delta - ck^{-1}b|$$

We have, on the set Q, $\Delta_j > 0$, $j = 1, 2, \ldots, n$ with $q_1^2 + q_2^2 + \ldots + q_n^2 \neq 0$. Consequently the vector function satisfies the conditions of the Chetaev theorem and this proves

Theorem 2. If the conditions: 1) function $V^\circ(q_1, q_2)$ satisfies the conditions of the Lemma and 2) function $V(q_1, q_2, \ldots, q_n)$ has no strictly local minimum of the position of equilibrium, both hold, then the position of equilibrium of the Hamiltonian system

$$q_j^\cdot = \frac{\partial H}{\partial p_j}, \quad p_j^\cdot = \frac{\partial H}{\partial q_j}, \quad j = 1, 2, \ldots, n$$

is unstable.

The author thanks V. V. Rumiantsev and the participants of his seminar for assessing the paper.

REFERENCES

1. Liapunov, A. M., General Problem of Stability of Motion. Ed. 2, Moscow — Leningrad, ONTI, 1935.

2. Chetaev, N. G., The Stability of Motion. (English translation), Pergamon Press. Book No. 09505, 1961.

3. Koiter, W. T., On the instability of equilibrium in the absence of a minimum of potential energy. Proc. Koninkl. Nederl. Acad. Wet. B. C, Vol. 68, 1965.

4. Hagedorn, P., Die Umkehrung der Stabilitäts-sätze von Lagrange — Dirichlet und Routh. - Orch. Rational Mech. and Analys., Vol. 42, No. 4, 1971.

5. Nemytskii, V. V. and Stepanov, V. V. Analytical Theory of Differential Equations. (English translation), Princeton University Press, Princeton, N. Y. 1960.

6. Erugin, N. P., Implicit Functions. Izd. LGU, Leningrad, 1956.

Translated by L.K.